

CENTRALLY SYMMETRIC CONFIGURATIONS OF ORDER POLYTOPES

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ABSTRACT. It is shown that the toric ideal of the centrally symmetric configuration of the order polytope of a finite partially ordered set possesses a squarefree quadratic initial ideal. It then follows that the convex polytope arising from the centrally symmetric configuration of an order polytope is a normal Gorenstein Fano polytope.

INTRODUCTION

Gorenstein Fano polytopes (reflexive polytopes) are interested in many researchers since they correspond to Gorenstein Fano varieties and are related with mirror symmetry. See, e.g., [4, §8.3] and its References. One of the most important problem is to find new classes of Gorenstein Fano polytopes. The centrally symmetric configuration [9] of an integer matrix supplies one of the most powerful tools to construct normal Gorenstein Fano polytopes. The purpose of the present paper is to study the centrally symmetric configuration of the integer matrix associated with the order polytope of a finite partially ordered set.

Let $\mathbb{Z}^{d \times n}$ denote the set of $d \times n$ integer matrices. Given $A \in \mathbb{Z}^{d \times n}$ for which no column vector is a zero vector, the *centrally symmetric configuration* of A is the $(d+1) \times (2n+1)$ integer matrix

$$A^\pm = \left[\begin{array}{c|cc} 0 & & \\ \vdots & A & -A \\ 0 & & \\ \hline 1 & 1 \ \dots \ 1 & 1 \ \dots \ 1 \end{array} \right].$$

On the other hand, the *centrally symmetric polytope* arising from A is the convex polytope $\mathcal{Q}_A^{(\text{sym})}$ which is the convex hull in \mathbb{R}^d of the column vectors of the matrix

$$\left[\begin{array}{c|c|c} 0 & & \\ \vdots & A & -A \\ 0 & & \end{array} \right].$$

We focus our attention on the problem when $\mathcal{Q}_A^{(\text{sym})}$ is a normal Gorenstein Fano polytope. In general, the origin is contained in the interior of $\mathcal{Q}_A^{(\text{sym})} \subset \mathbb{R}^d$. Suppose that $A \in \mathbb{Z}^{d \times n}$ satisfies $\mathbb{Z}A = \mathbb{Z}^d$. (Here $\mathbb{Z}A = \{z_1 \mathbf{a}_1 + \dots + z_n \mathbf{a}_n \mid z_1, \dots, z_n \in \mathbb{Z}\}$ for $A = [\mathbf{a}_1, \dots, \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$.) Then Lemma 1.1 in Section 1 guarantees that if the toric ideal I_{A^\pm} of A^\pm possesses a squarefree initial ideal with respect to a reverse

lexicographic order for which the variable corresponding to the column $[0, \dots, 0, 1]^t$ is smallest, then $\mathcal{Q}_A^{(\text{sym})}$ is a normal Gorenstein Fano polytope.

In [9], it is shown that if $\text{rank}(A) = d$ and all nonzero maximal minors of A are ± 1 , then $\mathcal{Q}_A^{(\text{sym})}$ is a normal Gorenstein Fano polytope. However, the converse does not hold. It is mentioned the existence of a matrix A such that A does not satisfy the above condition but $\mathcal{Q}_A^{(\text{sym})}$ is a normal Gorenstein Fano polytope ([9, Example 2.16]). Hence to construct an infinite family of matrices A like this is an important problem. The aim of this paper is to give an answer of this problem by using partially ordered sets.

Let P be a partially ordered set (poset) on $[d] = \{1, \dots, d\}$ and $\mathbf{e}_1, \dots, \mathbf{e}_d$ the unit coordinate vectors of \mathbb{R}^d . Given a subset $\alpha \subset P$, we write $\rho(\alpha) \in \mathbb{R}^d$ for the vector $\sum_{i \in \alpha} \mathbf{e}_i$. A *poset ideal* of P is a subset $\alpha \subset P$ such that if $a \in \alpha$ and $b \in P$ together with $b \leq a$, then $b \in \alpha$. In particular, the empty set as well as P itself is a poset ideal of P . Let $\mathcal{J}(P)$ denote the set of poset ideals of P . Note that $\mathcal{J}(P)$ has the structure of distributive lattice under inclusion, moreover, for any distributive lattice D , there exists a poset P such that $D \cong \mathcal{J}(P)$ by Birkhoff's Theorem [2]. The *order polytope* $\mathcal{O}(P)$ is the d -dimensional polytope which is the convex hull of $\{\rho(\alpha) \mid \alpha \in \mathcal{J}(P)\}$ in \mathbb{R}^d . See [11]. We then write A_P for the integer matrix whose column vectors are those $\rho(\alpha)^t$ with $\alpha \in \mathcal{J}(P) \setminus \{\emptyset\}$. This integer matrix A_P always satisfies $\mathbb{Z}A_P = \mathbb{Z}^d$, but does not always satisfy the condition that all nonzero maximal minors of A_P are ± 1 (Proposition 2.4). We prove that $\mathcal{Q}_{A_P}^{(\text{sym})}$ is a normal Gorenstein Fano polytope for *any* partially ordered set P .

The present paper is organized as follows. In Section 1, we recall basic materials on toric ideals of configurations as well as Fano polytopes. In Section 2, we will prove that, for an arbitrary finite partially ordered set P , the toric ideal $I_{A_P^\pm}$ of A_P^\pm possesses a squarefree quadratic initial ideal with respect to a reverse lexicographic order for which the variable corresponding to the column $[0, \dots, 0, 1]^t$ is smallest (Theorem 2.2). In particular, the centrally symmetric polytope $\mathcal{Q}_{A_P}^{(\text{sym})}$ arising from an arbitrary finite partially ordered set P is a normal Gorenstein Fano polytope (Corollary 2.3). Finally, in Section 3, we compute the δ -vectors of the centrally symmetric polytope $\mathcal{Q}_{A_P}^{(\text{sym})}$, where P is an antichain.

1. CENTRALLY SYMMETRIC CONFIGURATIONS AND FANO POLYTOPES

We recall fundamental materials on centrally symmetric configurations and Fano polytopes. Let, as before, $\mathbb{Z}^{d \times n}$ denote the set of $d \times n$ integer matrices.

a) Configuration

A matrix $A \in \mathbb{Z}^{d \times n}$ is called a *configuration* if there exists a hyperplane $\mathcal{H} \subset \mathbb{R}^d$ not passing the origin of \mathbb{R}^d such that each column vector of A lies on \mathcal{H} .

Let K be a field and $K[T^{\pm 1}] = K[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}]$ the Laurent polynomial ring in d variables over K . We associate each vector $\mathbf{a} = [a_1, \dots, a_d]^t \in \mathbb{Z}^d$, with the Laurent monomial $T^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d} \in K[T^{\pm 1}]$. Given a configuration $A \in \mathbb{Z}^{d \times n}$ with $\mathbf{a}_1, \dots, \mathbf{a}_n$ its column vectors, the *toric ring* $K[A]$ of A is the monomial subalgebra of $K[T^{\pm 1}]$ which is generated by $T^{\mathbf{a}_1}, \dots, T^{\mathbf{a}_n}$. Let $K[X] = K[x_1, \dots, x_n]$ denote the

polynomial ring in n variables over K . We define the surjective ring homomorphism $\pi : K[X] \rightarrow K[A]$ by setting $\pi(x_i) = T^{\mathbf{a}_i}$ for $i = 1, \dots, n$. The kernel I_A of π is called the *toric ideal* of A .

b) Fano polytope

A convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is called *integral* if each vertex of \mathcal{P} belongs to \mathbb{Z}^d . We say that an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is *normal* if, for each integer $N > 0$ and for each $\mathbf{a} \in N\mathcal{P} \cap \mathbb{Z}^d$, there exist $\mathbf{a}_1, \dots, \mathbf{a}_N$ belonging to $\mathcal{P} \cap \mathbb{Z}^d$ such that $\mathbf{a} = \mathbf{a}_1 + \dots + \mathbf{a}_N$. Now, an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is said to be a *Fano polytope* if the dimension of \mathcal{P} is d and if the origin of \mathbb{R}^d is the unique integer point belonging to the interior of \mathcal{P} . A Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ is called *Gorenstein* if its dual polytope

$$\mathcal{P}^\vee = \{ \mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{y} \in \mathcal{P} \}$$

is integral, where $\langle \mathbf{x}, \mathbf{y} \rangle$ is a canonical inner product of \mathbb{R}^d .

We now come to an essential lemma on normal Fano polytopes. We refer the reader to [14] and [6, Chapters 1 and 5] for basic information on initial ideals of toric ideals, regular and unimodular triangulations of integral convex polytopes.

Lemma 1.1. *Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope such that the origin is contained in its interior. Let $\mathcal{P} \cap \mathbb{Z}^d = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and*

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ 1 & \cdots & 1 \end{bmatrix} \in \mathbb{Z}^{(d+1) \times n}.$$

Suppose that $\mathbb{Z}A = \mathbb{Z}^{d+1}$ and that there exists an ordering of the variables $x_{i_1} < \dots < x_{i_n}$ for which $\mathbf{a}_{i_1} = \mathbf{0}$ such that the initial ideal $\text{in}_{<}(I_A)$ of the toric ideal I_A with respect to the reverse lexicographic order $<$ on $K[X]$ induced by the ordering is squarefree. Then \mathcal{P} is a normal Gorenstein Fano polytope.

Proof. The existence of an initial ideal as stated above guarantees that \mathcal{P} possesses a unimodular triangulation Δ (i.e., a triangulation Δ of \mathcal{P} such that the normalized volume of each maximal simplex in Δ is one) for which each maximal face of Δ contains the origin of \mathbb{R}^d as a vertex. See [14, Proposition 8.6 and Corollary 8.9]. It then follows easily that \mathcal{P} is Fano and that the supporting hyperplane of each facet of \mathcal{P} is of the form

$$\{[z_1, \dots, z_d]^t \in \mathbb{R}^d \mid a_1 z_1 + \dots + a_d z_d = 1\}$$

with each $a_i \in \mathbb{Z}$. Hence the dual polytope of \mathcal{P} is integral. Furthermore, in general, the existence of a unimodular triangulation of \mathcal{P} says that \mathcal{P} is normal (see [14, Proposition 13.15]). \square

c) Centrally symmetric configuration

In [9], the *centrally symmetric configuration* A^\pm of a matrix $A \in \mathbb{Z}^{d \times n}$ is introduced:

$$A^\pm = \left[\begin{array}{c|cc|cc} 0 & & & & & \\ \vdots & & A & & -A & \\ 0 & & & & & \\ \hline 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{array} \right].$$

A basic fact obtained in [9] is:

Proposition 1.2 ([9]). *Let $A \in \mathbb{Z}^{d \times n}$ be a matrix such that $\mathbb{Z}A = \mathbb{Z}^d$ and whose nonzero maximal minors are ± 1 . Then the toric ideal I_{A^\pm} of A^\pm possesses a square-free quadratic initial ideal with respect to a reverse lexicographic order for which the variable corresponding to the column $[0, \dots, 0, 1]^t$ is smallest. In particular, $\mathcal{Q}_A^{(\text{sym})}$ is a normal Gorenstein Fano polytope.*

Remark 1.3. In general, $A \in \mathbb{Z}^{d \times n}$ is called *unimodular* if $\text{rank}(A) = d$ and all nonzero maximal minors of A have the same absolute value. If we assume $\mathbb{Z}A = \mathbb{Z}^d$, then A is unimodular if and only if all nonzero maximal minors of A are ± 1 .

Example 1.4. Let A be the following configuration:

$$A = \left[\begin{array}{c|ccccccccc} 0 & & & & & & & & & \\ \vdots & & & & & & & & & \\ 0 & & & & & & & & & \\ \hline 1 & 1 & \dots & 1 & & & & & & \end{array} \right] \text{ where } A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then we have $\mathbb{Z}A = \mathbb{Z}^7$ but A is not unimodular. The initial ideal $\text{in}_{<}(I_A)$ of I_A with respect to any reverse lexicographic order $<$ is squarefree. Moreover, I_A has a quadratic Gröbner basis with respect to some reverse lexicographic order such that the smallest variable is x_1 . In addition, $K[A]$ is Gorenstein. On the other hand, $\mathcal{Q}_A^{(\text{sym})}$ is not normal. We can check that I_{A^\pm} is not generated by quadratic binomials and $\mathcal{Q}_A^{(\text{sym})}$ is not Gorenstein by using the software package CoCoA [3].

2. CENTRALLY SYMMETRIC CONFIGURATIONS OF ORDER POLYTOPES

In this section, we prove that, for any poset P ,

- the toric ideal $I_{A_P^\pm}$ possesses a squarefree quadratic initial ideal with respect to a reverse lexicographic order such that the smallest variable corresponds to the origin; (Theorem 2.2);
- $\mathcal{Q}_{A_P}^{(\text{sym})}$ is a normal Gorenstein Fano polytope (Corollary 2.3).

Let P be a poset on $[d] = \{1, \dots, d\}$ and let $K[s, T^{\pm 1}] = K[s, t_1, t_1^{-1}, \dots, t_d, t_d^{-1}]$ be a Laurent polynomial ring in $d+1$ variables over K . We set S_P to be the polynomial ring in variables z , x_I ($\emptyset \neq I \in \mathcal{J}(P)$), and y_I ($\emptyset \neq I \in \mathcal{J}(P)$) over K . We define the ring homomorphism $\pi : S_P \rightarrow K[s, T^{\pm 1}]$ by setting $\pi(z) = s$, $\pi(x_I) = s \prod_{i \in I} t_i$ and $\pi(y_I) = s \prod_{i \in I} t_i^{-1}$. Then the toric ideal $I_{A_P^\pm}$ is the kernel of π and the toric ring $K[A_P^\pm]$ is the image of π . Let $<$ be a reverse lexicographic order on S_P which satisfies $x_I < x_J$ and $y_I < y_J$ for all $I, J \in \mathcal{J}(P)$ with $I \subset J$. Here we set $x_\emptyset = y_\emptyset = z$ and hence z is the smallest variable in S_P .

Example 2.1. Let P be the poset on $\{1, 2, 3, 4, 5\}$ with the partial order $1 < 3, 2 < 3, 2 < 4$ and $4 < 5$. In this case, A_P is the following matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Then the initial ideal of $I_{A_P^\pm}$ with respect to a reverse lexicographic order $<$ is generated by 66 squarefree quadratic monomials.

Theorem 2.2. *Let \mathcal{G} be the set of binomials of the following types:*

$$\begin{aligned} x_I x_J - x_{I \cup J} x_{I \cap J}, \quad y_I y_J - y_{I \cup J} y_{I \cap J} \quad (I \not\sim J) \\ x_I y_J - x_{I \setminus k} y_{J \setminus k} \quad (k \text{ is a maximal element of both } I \text{ and } J). \end{aligned}$$

Then \mathcal{G} is a Gröbner bases of $I_{A_P^\pm}$ with respect to a reverse lexicographic order $<$, and the initial monomial of each binomial in \mathcal{G} is the first monomial.

Proof. Let $\text{in}(\mathcal{G}) = \langle \text{in}_<(g) \mid g \in \mathcal{G} \rangle$. Assume that \mathcal{G} is not a Gröbner bases of $I_{A_P^\pm}$. Then there exists a non-zero irreducible homogeneous binomial $f = u - v \in I_{A_P^\pm}$ such that neither u nor v belongs to $\text{in}(\mathcal{G})$. It then follows that there exist $\mathcal{I}, \mathcal{J}, \mathcal{I}'$, and \mathcal{J}' satisfying:

$$u = z^\alpha \prod_{I \in \mathcal{I}} x_I^{p_I} \prod_{J \in \mathcal{J}} y_J^{q_J}, \quad v = z^{\alpha'} \prod_{I' \in \mathcal{I}'} x_{I'}^{p_{I'}} \prod_{J' \in \mathcal{J}'} y_{J'}^{q_{J'}},$$

where $0 < p_I, q_J, p_{I'}, q_{J'} \in \mathbb{Z}$ for all $I \in \mathcal{I}, J \in \mathcal{J}, I' \in \mathcal{I}'$, and $J' \in \mathcal{J}'$.

Since $x_I y_I \nmid u, v$ for all $\emptyset \neq I \in \mathcal{J}(P)$, we have $\mathcal{I} \cap \mathcal{J} = \mathcal{I}' \cap \mathcal{J}' = \emptyset$. Moreover, since $x_I x_J, y_I y_J \nmid u, v$ for all $I \not\sim J$, we can see that $\mathcal{I}, \mathcal{J}, \mathcal{I}'$, and \mathcal{J}' are totally ordered sets. In addition, we have $\mathcal{I} \cap \mathcal{I}' = \mathcal{J} \cap \mathcal{J}' = \emptyset$ since f is irreducible.

Let $p = \sum_{I \in \mathcal{I}} p_I$, $q = \sum_{J \in \mathcal{J}} q_J$, $p' = \sum_{I' \in \mathcal{I}'} p_{I'}$ and $q' = \sum_{J' \in \mathcal{J}'} q_{J'}$. Then

$$\begin{aligned} \pi(u) &= s^{\alpha+p+q} \prod_{I \in \mathcal{I}} \left(\prod_{i \in I} t_i \right)^{p_I} \prod_{J \in \mathcal{J}} \left(\prod_{j \in J} t_j^{-1} \right)^{q_J}, \\ \pi(v) &= s^{\alpha'+p'+q'} \prod_{I' \in \mathcal{I}'} \left(\prod_{i \in I'} t_i \right)^{p_{I'}} \prod_{J' \in \mathcal{J}'} \left(\prod_{j \in J'} t_j^{-1} \right)^{q_{J'}}. \end{aligned}$$

Since $\pi(u) = \pi(v)$,

$$(1) \quad \sum_{\substack{I \in \mathcal{I} \\ a \in I}} p_I - \sum_{\substack{J \in \mathcal{J} \\ a \in J}} q_J = \sum_{\substack{I' \in \mathcal{I}' \\ a \in I'}} p_{I'} - \sum_{\substack{J' \in \mathcal{J}' \\ a \in J'}} q_{J'}$$

holds for all $a \in P$. In other words,

$$(2) \quad \sum_{\substack{I \in \mathcal{I} \\ a \in I}} p_I + \sum_{\substack{J' \in \mathcal{J}' \\ a \in J'}} q_{J'} = \sum_{\substack{I' \in \mathcal{I}' \\ a \in I'}} p_{I'} + \sum_{\substack{J \in \mathcal{J} \\ a \in J}} q_J$$

holds for all $a \in P$.

Let I_{\max} , I'_{\max} , J_{\max} and J'_{\max} be the maximal elements of \mathcal{I} , \mathcal{I}' , \mathcal{J} and \mathcal{J}' , respectively. (If $\mathcal{J} = \emptyset$, then let $J_{\max} = \emptyset$.) Since $x_{I_{\max}}y_{J_{\max}}$ does not belong to $\text{in}(\mathcal{G})$, I_{\max} and J_{\max} do not have a common maximal element. Note that the left (resp. right) side of equation (2) is not zero if and only if $a \in P$ belongs to $I_{\max} \cup J'_{\max}$ (resp. $I'_{\max} \cup J_{\max}$). Hence $I_{\max} \cup J'_{\max} = I'_{\max} \cup J_{\max}$ holds. Suppose that $I'_{\max} = \emptyset$. Then $I_{\max} \cup J'_{\max} = J_{\max}$. If $J_{\max} = \emptyset$, then we have $I_{\max} = I'_{\max} = J_{\max} = J'_{\max} = \emptyset$. Therefore $f = 0$, that is, $u = v$. This is a contradiction. If $J_{\max} \neq \emptyset$, then let $\{b_1, \dots, b_s\}$ be the set of maximal elements of J_{\max} . Then $J_{\max} = \bigcup_{i=1}^s \langle b_i \rangle$, where $\langle b_i \rangle = \{p \in P \mid p \leq b_i\}$. Since I_{\max} and J_{\max} do not share a common maximal element, $b_i \notin I_{\max}$ for all $1 \leq i \leq s$. Then $b_i \in J'_{\max}$ and we have $J_{\max} \subset J'_{\max}$. Hence $J'_{\max} \subset I_{\max} \cup J'_{\max} = J_{\max} \subset J'_{\max}$. Therefore we have $J_{\max} = J'_{\max}$, but this contradicts that $\mathcal{J} \cap \mathcal{J}' = \emptyset$. Hence $I'_{\max} \neq \emptyset$. Similarly, we have $I_{\max} \neq \emptyset$.

Since $\mathcal{I} \cap \mathcal{I}' = \emptyset$, we have $I_{\max} \neq I'_{\max}$. Hence we may assume that $I_{\max} \not\subset I'_{\max}$. Then there exists $a_{\max} \in I_{\max}$ such that a_{\max} is a maximal element of I_{\max} and $a_{\max} \notin I'_{\max}$. Then $a_{\max} \in J_{\max}$. Since a_{\max} is not a maximal element of J_{\max} , we have $\{c \in J_{\max} \mid c > a_{\max}\} \neq \emptyset$. Let $\{c_1, \dots, c_t\} = \{c \in J_{\max} \mid c > a_{\max}\}$.

For each $1 \leq j \leq t$, $c_j \notin I_{\max}$ since $c_j > a_{\max}$. Thus we have $c_j \notin I$ for all $I \in \mathcal{I} \cup \mathcal{I}'$ and $a_{\max} \notin I'$ for all $I' \in \mathcal{I}'$. Hence we have

$$-\sum_{\substack{J \in \mathcal{J} \\ c_j \in J}} q_J = -\sum_{\substack{J' \in \mathcal{J}' \\ c_j \in J'}} q'_{J'}, \quad \sum_{\substack{I \in \mathcal{I} \\ a_{\max} \in I}} p_I - \sum_{\substack{J \in \mathcal{J} \\ a_{\max} \in J}} q_J = -\sum_{\substack{J' \in \mathcal{J}' \\ a_{\max} \in J'}} q'_{J'}$$

by equation (1). Note that if $c_j \in J \in \mathcal{J}(P)$, then $a_{\max} \in J$. Thus we have

$$\sum_{\substack{I \in \mathcal{I} \\ a_{\max} \in I}} p_I - \sum_{\substack{J \in \mathcal{J} \\ a_{\max} \in J, c_j \notin J}} q_J = -\sum_{\substack{J' \in \mathcal{J}' \\ a_{\max} \in J', c_j \notin J'}} q'_{J'} \leq 0.$$

Since

$$\sum_{\substack{I \in \mathcal{I} \\ a_{\max} \in I}} p_I \geq p_{I_{\max}} > 0,$$

there exists $J_j \in \mathcal{J}$ such that $a_{\max} \in J_j$ and $c_j \notin J_j$ for each $1 \leq j \leq t$.

We may assume that $J_1 \subset \dots \subset J_t (\subset J_{\max})$ since \mathcal{J} is a totally ordered set. Then $J_1 \cap \{c_1, \dots, c_t\} = \emptyset$. Thus a_{\max} is a maximal element of both I_{\max} and J_1 , and hence $x_{I_{\max}}y_{J_1}$ belongs to $\text{in}(\mathcal{G})$. This is a contradiction. Therefore \mathcal{G} is a Gröbner basis of $I_{A_P}^{\pm}$. \square

In Theorem 2.2, the initial ideal is squarefree and quadratic. By Lemma 1.1, we have the following:

Corollary 2.3. *Let P be a poset. Then $\mathcal{Q}_{A_P}^{(\text{sym})}$ is a normal Gorenstein Fano polytope.*

Note that the matrix A_P of a poset P is not necessarily unimodular. A characterization of the unimodularity of A_P appeared in [8, Example 3.6] without proof. Here, we recall the characterization with a proof. Let \mathbb{N}^2 be the distributive lattice consisting of all pairs (i, j) of nonnegative integers with the partial order $(i, j) \leq (k, \ell)$ if and only if $i \leq k$ and $j \leq \ell$. A distributive lattice D is said to be *planar* (see [1, p. 436]) if D is a finite sublattice of \mathbb{N}^2 with $(0, 0) \in D$ which satisfies the following:

for any $(i, j), (k, \ell) \in D$ with $(i, j) < (k, \ell)$, there exists a chain (totally ordered subset) of D of the form $(i, j) = (i_0, j_0) < (i_1, j_1) < \cdots < (i_s, j_s) = (k, \ell)$ such that $i_{k+1} + j_{k+1} = i_k + j_k + 1$ for all k .

Proposition 2.4 ([8]). *Let $\mathcal{J}(P)$ be the distributive lattice associated with a poset P . Then A_P is unimodular if and only if $\mathcal{J}(P)$ is planar.*

Proof. Assume that $\mathcal{J}(P)$ is planar. Then $K[A_P]$ is isomorphic to $K[A]$ where A is the vertex-edge incidence matrix of a bipartite graph (see [7]). Since A is unimodular [10], A_P is also unimodular.

Conversely, assume that $\mathcal{J}(P)$ is not planar. Then there exist $a, b, c \in P$ such that $a \not\prec b$, $a \not\prec c$ and $b \not\prec c$. Let $I = \langle a \rangle \cup \langle b \rangle \cup \langle c \rangle$. Then the sublattice $\{J \in \mathcal{J}(P) \mid I \setminus \{a, b, c\} \subset J \subset I\}$ of $\mathcal{J}(P)$ is isomorphic to $\mathcal{J}(P')$, where $P' = \{a, b, c\}$ is an antichain. It is easy to see that

$$A_{P'} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

is not unimodular. Since $K[A_{P'}]$ is a combinatorial pure subring (see [8]) of $K[A_P]$, A_P is not unimodular. \square

3. THE δ -VECTOR OF $\mathcal{Q}_{A_P}^{(\text{sym})}$ OF AN ANTICHAIN POSET P

In this section, we show that the δ -vector of $\mathcal{Q}_{A_P}^{(\text{sym})}$ of an antichain poset P is the Eulerian number (see [13]).

First, we review Ehrhart theory on integral convex polytopes. Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d . Given integers $t = 1, 2, \dots$, we set $i(\mathcal{P}, t) = \#(t\mathcal{P} \cap \mathbb{Z}^d)$. Then $i(\mathcal{P}, t)$ is a polynomial in t of degree d and is called the *Ehrhart polynomial* of \mathcal{P} . Its generating function satisfies

$$1 + \sum_{t=1}^{\infty} i(\mathcal{P}, t) \lambda^t = \frac{\delta_{\mathcal{P}}(\lambda)}{(1 - \lambda)^{d+1}}$$

where $\delta_{\mathcal{P}}(\lambda) = \sum_{i=0}^d \delta_i \lambda^i$ is a polynomial in λ of degree $\leq d$. The vector $(\delta_0, \dots, \delta_d)$ is called the δ -vector of \mathcal{P} . It is known that a Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ is Gorenstein if and only if $\delta_i = \delta_{d-i}$ for all $0 \leq i \leq d$. See [5].

Let P be an antichain poset on $[d]$. One of the referees pointed out that $\mathcal{Q}_{A_P}^{(\text{sym})}$ is the Minkowski sum $C_d + L$ where C_d is the unit d -cube in \mathbb{R}^d and L is the closed line segment whose end points are the origin and the vector $\mathbf{v} = [-1, \dots, -1]^t \in \mathbb{R}^d$. In order to see this fact, note that $\mathcal{Q}_{A_P}^{(\text{sym})}$ is the convex hull of $C_d \cup (-C_d)$ and that $-C_d$ is the Minkowski sum $C_d + \mathbf{v}$. Hence $\mathcal{Q}_{A_P}^{(\text{sym})}$ is a *zonotope*, that is, a Minkowski sum of closed line segments:

$$\mathcal{Q}_{A_P}^{(\text{sym})} = \{r_1 \mathbf{e}_1 + \cdots + r_d \mathbf{e}_d + r_{d+1} \mathbf{v} \mid 0 \leq r_i \leq 1 \ (i = 1, 2, \dots, d+1)\}.$$

By using [12, Theorem 2.2], we have the following:

Proposition 3.1. *Let P be an antichain poset on $[d]$. Then we have*

$$\begin{aligned} i(\mathcal{Q}_{A_P}^{(\text{sym})}, t) &= (t+1)^{d+1} - t^{d+1}, \\ 1 + \sum_{t=1}^{\infty} i(\mathcal{Q}_{A_P}^{(\text{sym})}, t) \lambda^t &= \frac{\sum_{i=0}^d A(d+1, i) \lambda^i}{(1-\lambda)^{d+1}}, \end{aligned}$$

where $A(d+1, i)$ is the Eulerian number.

Proof. Let $B = \{\mathbf{e}_1, \dots, \mathbf{e}_d, \mathbf{v}\}$. Then B is not linearly independent and any proper subset of B is linearly independent. It is easy to see that any nonzero minor of the matrix $[\mathbf{e}_1, \dots, \mathbf{e}_d, \mathbf{v}] \in \mathbb{Z}^{d \times (d+1)}$ is ± 1 . Hence by [12, Theorem 2.2], we have

$$i(\mathcal{Q}_{A_P}^{(\text{sym})}, t) = \sum_{X \subsetneq B} t^{\#X} = (t+1)^{d+1} - t^{d+1}.$$

By a well-known identity

$$\sum_{t=1}^{\infty} t^d \lambda^t = \frac{\sum_{i=0}^d A(d, i) \lambda^{i+1}}{(1-\lambda)^{d+1}}$$

for the Eulerian number, it is easy to show

$$1 + \sum_{t=1}^{\infty} ((t+1)^{d+1} - t^{d+1}) \lambda^t = \frac{\sum_{i=0}^d A(d+1, i) \lambda^i}{(1-\lambda)^{d+1}}.$$

□

Remark 3.2. Let P be an antichain poset on $[d]$. It is known that

$$1 + \sum_{t=1}^{\infty} i(\mathcal{O}(P), t) \lambda^t = \frac{\sum_{i=0}^d A(d, i) \lambda^i}{(1-\lambda)^{d+1}}.$$

Example 3.3. Let P be the poset as appeared in Example 2.1. Then the δ -vector of $\mathcal{Q}_{A_P}^{(\text{sym})}$ is $(1, 15, 54, 54, 15, 1)$. Note that $\mathcal{O}(P)$ is not Gorenstein since P is not pure, and its δ -vector is $(1, 5, 3)$.

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